Feuille de calcul n°10 — Nombres complexes

Exercice 1. Écrire chacun des nombres complexes suivants sous forme algébrique.

$$z_1 = (2+6\,\mathrm{i})(5+\mathrm{i}) \qquad z_2 = (3-\mathrm{i})(4+\mathrm{i}) \qquad z_3 = (2-3\,\mathrm{i})^4 \qquad z_4 = \frac{1}{3-\mathrm{i}}$$

Solution.
$z_1 = 10 + 2\,\mathrm{i} + 30\,\mathrm{i} + 6\,\mathrm{i}^2 = 10 + 32\,\mathrm{i} - 6 = 4 + 32\,\mathrm{i}$
$z_2 = 12 + 3\,\mathrm{i} - 4\,\mathrm{i} - \mathrm{i}^2 = 12 - \mathrm{i} + 1 = 13 - \mathrm{i}$
$z_3 = (2 + (-3\,\mathrm{i}))^4 = \left[(2-3\,\mathrm{i})^2\right]^2 = (4 - 12\,\mathrm{i} + 9\,\mathrm{i}^2)^2 = (-5-12\,\mathrm{i})^2$
$\quad = (-5)^2 - 2\times(-5)\times 12\,\mathrm{i} + (12\,\mathrm{i})^2 = 25 + 120\,\mathrm{i} - 144 = -119 + 120\,\mathrm{i}$
$z_4 = \dfrac{3+\mathrm{i}}{3^2+1^2} = \dfrac{3+\mathrm{i}}{10} = \dfrac{3}{10} + \dfrac{1}{10}\,\mathrm{i}$

Exercice 2. Écrire chacun des nombres complexes suivants sous forme algébrique.

$$z_1 = (4-3\,\mathrm{i})^2 \qquad z_2 = (1-2\,\mathrm{i})(1+2\,\mathrm{i}) \qquad z_3 = \frac{2-3\,\mathrm{i}}{5+2\,\mathrm{i}} \qquad z_4 = \mathrm{e}^{-\mathrm{i}\frac{\pi}{3}}$$

Solution.
$z_1 = 16 - 24\,\mathrm{i} + 9\,\mathrm{i}^2 = 7 - 24\,\mathrm{i}$
$z_2 = 1^2 + 2^2 = 5$
$z_3 = \dfrac{(2-3\,\mathrm{i})(5-2\,\mathrm{i})}{5^2+2^2} = \dfrac{10 - 4\,\mathrm{i} - 15\,\mathrm{i} + 6\,\mathrm{i}^2}{29} = \dfrac{10 - 19\,\mathrm{i} - 6}{29} = \dfrac{4}{29} - \dfrac{19}{29}\,\mathrm{i}$
$z_4 = \cos\left(-\dfrac{\pi}{3}\right) + \mathrm{i}\sin\left(-\dfrac{\pi}{3}\right) = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}\,\mathrm{i}$

Exercice 3. Écrire les nombres complexes suivants sous forme exponentielle.

$$z_1 = 12 \qquad z_2 = -8 \qquad z_3 = \mathrm{i}\sqrt{3} \qquad z_4 = -2\,\mathrm{i}$$

Solution.
$z_1 = 12\,\mathrm{e}^{\mathrm{i}\,0}$
$z_2 = 8\times(-1) = 8\,\mathrm{e}^{\mathrm{i}\pi}$
$z_3 = \sqrt{3}\times\mathrm{i} = \sqrt{3}\,\mathrm{e}^{\mathrm{i}\frac{\pi}{2}}$
$z_4 = 2\times(-\mathrm{i}) = 2\,\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}}$

Exercice 4. Écrire les nombres complexes suivants sous forme exponentielle.

$$z_1 = -2\,\mathrm{e}^{\mathrm{i}\frac{3\pi}{5}} \qquad z_2 = 5 - 5\,\mathrm{i} \qquad z_3 = -5 + 5\,\mathrm{i}\sqrt{3} \qquad z_4 = \mathrm{e}^{\mathrm{i}\frac{\pi}{3}} + \mathrm{e}^{\mathrm{i}\frac{\pi}{6}}$$

Solution.

$z_1 = 2\,e^{\,i\left(\frac{3\pi}{5}+\pi\right)} = 2\,e^{\,i\,\frac{8\pi}{5}}$

$|z_2| = \sqrt{5^2+(-5)^2} = \sqrt{2\times 5^2} = 5\sqrt{2}$ donc $z_2 = 5\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\,\frac{1}{\sqrt{2}}\right) = 5\sqrt{2}\left(\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right)$ i.e. $z_2 = 5\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$ soit $z_2 = 5\sqrt{2}\,e^{-i\,\frac{\pi}{4}}$.

$|z_3| = \sqrt{(-5)^2+(5\sqrt{3})^2} = \sqrt{25+25\times 3} = \sqrt{100} = 10$ donc $z_3 = 10\left(-\frac{1}{2} + i\,\frac{\sqrt{3}}{2}\right)$ i.e. $z_3 = 10\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right)$ i.e. $z_3 = 10\,e^{\,i\,\frac{2\pi}{3}}$.

$z_4 = e^{\,i\,\frac{\pi}{4}}\left(e^{\,i\,\frac{\pi}{12}} + e^{-i\,\frac{\pi}{12}}\right) = e^{\,i\,\frac{\pi}{4}}\times 2\cos\left(\frac{\pi}{12}\right) = 2\cos\left(\frac{\pi}{12}\right)e^{\,i\,\frac{\pi}{4}}$. (Ceci est bien la forme algébrique car $0 \leqslant \frac{\pi}{12} \leqslant \frac{\pi}{2}$ donc $\cos\left(\frac{\pi}{12}\right) \geqslant 0$).

Exercice 5. On considère le nombre complexe $z = \dfrac{1+\sqrt{2}+i}{1+\sqrt{2}-i}$.

1. Calculer $|z|$.
2. Écrire z sous forme algébrique.
3. Déterminer la forme algébrique z^{2021}.

Solution.

1. Comme $\overline{1+\sqrt{2}+i} = 1+\sqrt{2}-i$, $\left|1+\sqrt{2}+i\right| = 1+\sqrt{2}-i$ et donc $|z| = \dfrac{\left|1+\sqrt{2}+i\right|}{\left|1+\sqrt{2}-i\right|} = 1$

2. $$\begin{aligned} z &= \frac{(1+\sqrt{2}+i)^2}{(1+\sqrt{2})^2+1^2} = \frac{(1+\sqrt{2})^2+2(1+\sqrt{2})\,i+i^2}{1+2\sqrt{2}+2+1} = \frac{1+2\sqrt{2}+2+2(1+\sqrt{2})\,i-1}{4+2\sqrt{2}} \\ &= \frac{2+2\sqrt{2}+2(1+\sqrt{2})\,i}{4+2\sqrt{2}} = \frac{2(1+\sqrt{2})}{4+2\sqrt{2}}(1+i) = \frac{1+\sqrt{2}}{2+\sqrt{2}}(1+i) \\ &= \frac{(1+\sqrt{2})(2-\sqrt{2})}{2^2-\sqrt{2}^2}(1+i) = \frac{2-\sqrt{2}+2\sqrt{2}-2}{4-2}(1+i) \\ &= \frac{\sqrt{2}}{2}(1+i) \end{aligned}$$
   donc la forme algébrique de z est $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\,i$.

3. $z = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right) = e^{\,i\,\frac{\pi}{4}}$ donc
   $$\begin{aligned} z^{2021} &= \left(e^{\,i\,\frac{\pi}{4}}\right)^{2021} = e^{\,i\,\frac{2021\pi}{4}} = e^{\,i\,\frac{2021\pi}{4}} = e^{\,i\left(\frac{5\pi}{4}+252\times 2\pi\right)} = e^{\,i\,\frac{5\pi}{4}} = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right) \\ &= -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\,i \end{aligned}$$