Feuille de calcul n°7 — Racines carrées

Exercice 1. Écrire chacun des nombres suivants sous la forme $a\sqrt{b}$ où a et b sont des entiers avec b le plus petit possible.

$$A = \sqrt{1000} \qquad B = \sqrt{125} \qquad C = \sqrt{27} \qquad D = \sqrt{27^3} \qquad E = (\sqrt{8})^5$$

Solution.
$A = \sqrt{100 \times 10} = \sqrt{100} \times \sqrt{10}$ donc $\boxed{A = 10\sqrt{10}}$.
$B = \sqrt{25 \times 5} = \sqrt{25} \times \sqrt{5}$ donc $\boxed{B = 5\sqrt{5}}$.
$C = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3}$ donc $\boxed{C = 3\sqrt{3}}$.
$D = \sqrt{27^2 \times 27} == \sqrt{27^2} \times \sqrt{27} = 27 \times 3\sqrt{3}$ donc $\boxed{D = 81\sqrt{3}}$.
$E = \sqrt{8}^4 \times \sqrt{8} = \left(\sqrt{8}^2\right)^2 \times \sqrt{4 \times 2} = 8^2 \times \sqrt{4} \times \sqrt{2} = 64 \times 2 \times \sqrt{2}$ donc $\boxed{E = 128\sqrt{2}}$.

Exercice 2. Écrire chacun des nombres suivants sans utiliser le symbole $\sqrt{\ }$.

$$A = \sqrt{5}\sqrt{45} \qquad B = \sqrt{(-1)^4} \qquad C = \sqrt{\frac{9}{25}} \qquad D = \frac{\sqrt{50}}{\sqrt{8}} \qquad E = \sqrt{8}\sqrt{162}$$

Solution.
$A = \sqrt{5} \times \sqrt{5 \times 9} = \sqrt{5} \times \sqrt{5} \times \sqrt{9} = \sqrt{5}^2 \times 3 = 5 \times 3$ donc $\boxed{A = 15}$.
$B = \sqrt{1}$ donc $\boxed{B = 1}$.
$C = \dfrac{\sqrt{9}}{\sqrt{25}}$ donc $\boxed{C = \dfrac{3}{5}}$.
$D = \dfrac{\sqrt{25 \times 2}}{\sqrt{4 \times 2}} = \dfrac{\sqrt{25} \times \sqrt{2}}{\sqrt{4} \times \sqrt{2}} = \dfrac{5\sqrt{2}}{2\sqrt{2}}$ donc $\boxed{D = \dfrac{5}{2}}$.
$E = \sqrt{4 \times 2}\sqrt{81 \times 2} = \sqrt{4} \times \sqrt{2} \times \sqrt{81} \times \sqrt{2} = 2\sqrt{2} \times 9\sqrt{2} = 18\sqrt{2}^2 = 18 \times 2$ donc $\boxed{E = 36}$.

Exercice 3. Écrire chacun des nombres suivants sans racine carrée au dénominateur.

$$A = \frac{2 - \sqrt{3}}{2 + \sqrt{3}} \qquad B = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} \qquad C = \frac{1}{\sqrt{2} - \sqrt{3}} \qquad D = \frac{\sqrt{2} + \sqrt{3}}{1 - \sqrt{3}}$$

Solution.
$A = \dfrac{(2 - \sqrt{3})(2 - \sqrt{3})}{(2 + \sqrt{3})(2 - \sqrt{3})} = \dfrac{(2 - \sqrt{3})^2}{2^2 - \sqrt{3}^2} = \dfrac{2^2 - 2 \times 2 \times \sqrt{3} + \sqrt{3}^2}{4 - 3} = \dfrac{4 - 4\sqrt{3} + 3}{1}$ donc $\boxed{A = 7 - 4\sqrt{3}}$.
$B = \dfrac{(\sqrt{2} - 1)(\sqrt{2} - 1)}{(\sqrt{2} + 1)(\sqrt{2} - 1)} = \dfrac{(\sqrt{2} - 1)^2}{\sqrt{2}^2 - 1^2} = \dfrac{\sqrt{2}^2 - 2 \times 1 \times \sqrt{2} + 1^2}{2 - 1} = \dfrac{2 - 2\sqrt{3} + 1}{1}$ donc $\boxed{B = 3 - 2\sqrt{2}}$.

$C = \dfrac{1 \times (\sqrt{2}+\sqrt{3})}{(\sqrt{2}-\sqrt{3})(\sqrt{2}+\sqrt{3})} = \dfrac{\sqrt{2}+\sqrt{3}}{\sqrt{2}^2-\sqrt{3}^2} = \dfrac{\sqrt{2}+\sqrt{3}}{2-3} = \dfrac{\sqrt{2}+\sqrt{3}}{-1}$ donc $\boxed{C = -\sqrt{2}-\sqrt{3}}$.

$D = \dfrac{(\sqrt{2}+\sqrt{3})(1+\sqrt{3})}{(1-\sqrt{3})(1+\sqrt{3})} = \dfrac{\sqrt{2}+\sqrt{6}+\sqrt{3}+\sqrt{3}^2}{1^2-\sqrt{3}^2} = \dfrac{\sqrt{2}+\sqrt{6}+\sqrt{3}+3}{1-3} = \dfrac{3+\sqrt{2}+\sqrt{3}+\sqrt{6}}{-2}$

donc $\boxed{D = -\dfrac{3+\sqrt{2}+\sqrt{3}+\sqrt{6}}{2}}$.

Exercice 4. Écrire chacun des nombres suivants sans racine carrée au dénominateur.

$$A = \frac{\sqrt{5}-\sqrt{2}}{\sqrt{3}-\sqrt{2}} \qquad B = \frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}} \qquad C = \frac{5+2\sqrt{6}}{\sqrt{2}+\sqrt{3}} + \frac{5-2\sqrt{6}}{\sqrt{2}-\sqrt{3}} \qquad D = \left(\frac{5\sqrt{2}}{1-\sqrt{3}}\right)^2$$

Solution.

$A = \dfrac{(\sqrt{5}-\sqrt{2})(\sqrt{3}+\sqrt{2})}{(\sqrt{3}-\sqrt{2})(\sqrt{3}+\sqrt{2})} = \dfrac{\sqrt{15}+\sqrt{10}-\sqrt{6}-2}{\sqrt{3}^2-\sqrt{2}^2} = \dfrac{\sqrt{15}+\sqrt{10}-\sqrt{6}-2}{3-2}$ et donc on conclut que $\boxed{A = \sqrt{15}+\sqrt{10}-\sqrt{6}-2}$.

$$B = \frac{(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} = \frac{2-\sqrt{6}+\sqrt{6}-3+\sqrt{10}-\sqrt{15}}{\sqrt{2}^2-\sqrt{3}^2}$$
$$= \frac{-1+\sqrt{10}-\sqrt{15}}{2-3} = \frac{-1+\sqrt{10}-\sqrt{15}}{-1}$$

donc $\boxed{B = 1-\sqrt{10}+\sqrt{15}}$.

$$C = \frac{(5+2\sqrt{6})(\sqrt{2}-\sqrt{3})}{(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})} + \frac{(5-2\sqrt{6})(\sqrt{2}+\sqrt{3})}{(\sqrt{2}-\sqrt{3})(\sqrt{2}+\sqrt{3})}$$
$$= \frac{5\sqrt{2}-5\sqrt{3}+2\sqrt{12}-2\sqrt{18}+5\sqrt{2}+5\sqrt{3}-2\sqrt{12}-2\sqrt{18}}{\sqrt{2}^2-\sqrt{3}^2}$$
$$= \frac{10\sqrt{2}-4\sqrt{18}}{2-3} = \frac{10\sqrt{2}-4\sqrt{9\times 2}}{-1} = \frac{10\sqrt{2}-12\sqrt{2}}{-1}$$

donc $\boxed{C = 2\sqrt{2}}$.

$$D = \frac{(5\sqrt{2})^2}{(1-\sqrt{3})^2} = \frac{5^2\sqrt{2}^2}{1-2\sqrt{3}+\sqrt{3}^2} = \frac{50}{4-2\sqrt{3}} = \frac{25}{2-\sqrt{3}} = \frac{25(2+\sqrt{3})}{(2-\sqrt{3})(2+\sqrt{3})}$$
$$= \frac{50+25\sqrt{3}}{2^2-\sqrt{3}^2}$$

donc $\boxed{D = 50+25\sqrt{3}}$.

Exercice 5. Écrire les nombres suivants sous la forme $a\sqrt{b}$ où a et b sont des entiers avec b le plus petit possible.

$$A = \sqrt{2}+\sqrt{8} \qquad B = \sqrt{27}+\sqrt{48} \qquad C = \sqrt{44}+\sqrt{99}+\sqrt{1100} \qquad D = 5\sqrt{20}+2\sqrt{45}-3\sqrt{80}$$

Solution.

$A = \sqrt{2}+\sqrt{4\times 2} + \sqrt{2}+2\sqrt{2}$ donc $\boxed{A = 3\sqrt{2}}$.

$B = \sqrt{9\times 3}+\sqrt{16\times 3} = 3\sqrt{3}+4\sqrt{3}$ donc $\boxed{B = 7\sqrt{3}}$.

$C = \sqrt{4\times 11}+\sqrt{9\times 11}+\sqrt{100\times 11} = 2\sqrt{11}+3\sqrt{11}+10\sqrt{11}$ donc $\boxed{C = 15\sqrt{11}}$.

$D = 5\sqrt{4\times 5}+2\sqrt{9\times 5}-3\sqrt{16\times 5} = 5\times 2\sqrt{5}+2\times 3\sqrt{5}-3\times 4\sqrt{5} = 10\sqrt{5}+6\sqrt{5}-12\sqrt{5}$ donc $\boxed{D = 4\sqrt{5}}$.

Exercice 6. Soit $a \in \left]0\,;\frac{1}{2}\right]$. Montrer que

$$\frac{1+\sqrt{1-4a^2}}{2a} = \frac{2a}{1-\sqrt{1-4a^2}}.$$

Solution.

$$\begin{aligned}
\frac{2a}{1-\sqrt{1-4a^2}} &= \frac{2a\left(1+\sqrt{1-4a^2}\right)}{\left(1-\sqrt{1-4a^2}\right)\left(1+\sqrt{1-4a^2}\right)}\\
&= \frac{2a\left(1+\sqrt{1-4a^2}\right)}{1^2-\sqrt{1-4a^2}^2}\\
&= \frac{2a\left(1+\sqrt{1-4a^2}\right)}{1-(1-4a^2)}\\
&= \frac{2a\left(1+\sqrt{1-4a^2}\right)}{1-1+4a^2}\\
&= \frac{2a\left(1+\sqrt{1-4a^2}\right)}{(2a)^2}
\end{aligned}$$

donc, en simplifiant par $2a$,

$$\boxed{\frac{1+\sqrt{1-4a^2}}{2a} = \frac{2a}{1-\sqrt{1-4a^2}}}.$$